

ON THE ZEROS OF SUCCESSIVE DERIVATIVES OF INTEGRAL FUNCTIONS

BY

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1. The Gontcharoff polynomials

$$G_0(z) = 1; \quad G_n(z; z_1, z_2, \dots, z_n) = \int_{z_1}^z dz' \int_{z_2}^{z'} dz'' \cdots \int_{z_n}^{z^{(n-1)}} dz^{(n)} \quad (n \geq 1)$$

have applications to a certain class of interpolation problem (Whittaker [7])(¹). In this paper I obtain some formulae connected with these polynomials and use them to improve and extend a theorem due to Levinson [3, 4], and to shorten the proof of and extend a theorem due to Schoenberg [6].

LEVINSON'S THEOREM. *If $f(z)$ is an integral function satisfying*

$$\limsup_{r \rightarrow \infty} \frac{\log M(r)}{r} < .7199,$$

and if $f(z)$ and each of its derivatives have at least one zero in or on the unit circle, then $f(z) \equiv 0$.

The constant .7199 is not the "best possible" but cannot be replaced [5] by a number as great as .7378.

The "best possible" value of this constant is known as the Whittaker constant W . Among new results in this paper, I prove that W cannot be less than .7259.

SCHOENBERG'S THEOREM. *If $f(z)$ is an integral function satisfying*

$$\limsup_{r \rightarrow \infty} \frac{\log M(r)}{r} < \frac{\pi}{4},$$

and if $f(z)$ and each of its derivatives have at least one zero in the segment $-1 \leq x \leq 1$ of the real axis, then $f(z) \equiv 0$.

The constant $\pi/4$ is the "best possible" as shown by the example $\cos(\pi z/4) + \sin(\pi z/4)$.

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(¹) Numbers in brackets refer to the references cited at the end of the paper.

2. Following the notation used by Levinson [3], let

$$\begin{aligned} H_0(z) &= 1; \quad H_n(z_1, z_2, \dots, z_n) = G_n(0; z_1, z_2, \dots, z_n) \quad (n \geq 1), \\ M_n &= \max |G_n(z_0; z_1, z_2, \dots, z_n)| \quad (\text{all } |z_r| \leq 1), \\ L_n &= \max |H_n(z_1, z_2, \dots, z_n)| \quad (\text{all } |z_r| \leq 1). \end{aligned}$$

We first require two inequalities (2.1) and (2.4) due to Levinson and (for the sake of completeness) give his proof. Since by definition

$$G_n(z_0; z_1, \dots, z_n) = H_n(z_1, z_2, \dots, z_n) - H_n(z_0, z_2, \dots, z_n),$$

therefore, by Taylor's Theorem

$$\begin{aligned} G_n(z_0; z_1, z_2, \dots, z_n) &= \sum_{r=1}^n \frac{(z_1)^r}{r!} H_{n-r}(z_{r+1}, z_{r+2}, \dots, z_n) \\ &\quad - \sum_{r=1}^n \frac{(z_0)^r}{r!} H_{n-r}(z_{r+1}, z_{r+2}, \dots, z_n). \end{aligned}$$

Hence

$$|G_n(z_0; z_1, \dots, z_n)| \leq \sum_{r=1}^n \frac{|z_1^r - z_0^r|}{r!} L_{n-r}$$

and, if we write $2\alpha = \arg z_1 - \arg z_0$,

$$(2.1) \quad M_n \leq \max_{0 \leq \alpha \leq \pi/2} \left\{ \sum_{r=1}^n \frac{2 |\sin r\alpha|}{r!} L_{n-r} \right\}.$$

By Euler's formula for homogeneous functions,

$$nG_n = \sum_{r=0}^n z_r \frac{\partial G_n}{\partial z_r},$$

and since

$$(2.2) \quad \frac{\partial G_n}{\partial z_0} = G_{n-1}(z_0; z_2, \dots, z_n),$$

$$(2.3) \quad \frac{\partial G_n}{\partial z_r} = -G_{r-1}(z_0; z_1, \dots, z_{r-1}) \times G_{n-r}(z_r; z_{r+1}, \dots, z_n) \quad (r \geq 1),$$

we have the inequality

$$(2.4) \quad nM_n \leq M_{n-1} + \sum_{r=1}^n M_{r-1} M_{n-r}.$$

It is obvious, as Levinson points out, that $L_1=1$, $L_2=3/2$, $M_1=2$, and hence from (2.1) he obtains $M_2 \leq (3/2)3^{1/2} < 2.5981$, $M_3 < 3.6379$. By special

choice of the z_r he shows that these values are "accurate" and that in fact $M_2 = (3/2)3^{1/2}$ and $M_3 > 3.6378$. It can also be proved [4] that $L_3 = 2^{-1}[2(5)^{1/2} + 3]^{1/2} + 6^{-1}[6(5)^{1/2} - 2]^{1/2} < 1.9299$, and again, by use of (2.1) he obtains [4] $M_4 < 4.8414$. He then uses (2.4) to find upper bounds for M_5 , M_6 , M_7 , M_8 , M_9 , and (by induction) M_n . In fact $M_n \leq r^{n+1}$ ($n > 1$) where $r < 1.389$. He remarks that this method would presumably yield a better value of r if accurate values of some further members of the sequence M_n were worked out before resorting to the use of formula (2.4). However the problem of determining L_4 or M_5 exactly is not simple and for higher L_n , M_n , this does not seem a very promising line of approach.

3. It is, however, possible to obtain upper bounds for L_4 , and so on, by using another iteration formula involving both sequences L_n and M_n . For Euler's formula gives

$$nH_n = \sum_{r=1}^n z_r \frac{\partial H_n}{\partial z_r}$$

and since

$$\partial H_n / \partial z_r = -H_{r-1}(z_1; z_2, \dots, z_{r-1}) \times G_{n-r}(z_r; z_{r+1}, \dots, z_n),$$

we have the inequality

$$(3.1) \quad nL_n \leq \sum_{r=1}^n L_{r-1}M_{n-r}.$$

In particular, when $n=4$, $4L_4 \leq L_0M_3 + L_1M_2 + L_2M_1 + L_3M_0$, yielding $L_4 < 2.7915$, and (2.1) gives $M_5 \leq \max_{0 \leq \alpha \leq \pi/2} \phi_5(\alpha)$ where

$$\begin{aligned} \phi_5(\alpha) = & 5.5830 |\sin \alpha| + 1.9299 |\sin 2\alpha| + (1/2) |\sin 3\alpha| \\ & + (1/12) |\sin 4\alpha| + (1/60) |\sin 5\alpha|. \end{aligned}$$

The maximum on this curve lies between $70^\circ 27'$ and $70^\circ 28'$ and shows that $M_5 < 6.8223$.

Proceeding in this way by alternate use of (2.1) and (3.1), we find upper bounds for L_5 , L_6 , L_7 , L_8 , L_9 , L_{10} ; M_6 , M_7 , M_8 , M_9 , and M_{10} (see appendix). The curves whose maxima have to be determined may be taken as

$$\begin{aligned} \phi_6(\alpha) = & 7.6112 |\sin \alpha| + 2.7915 |\sin 2\alpha| + 0.6433 |\sin 3\alpha| \\ & + (1/8) |\sin 4\alpha| + (1/60) |\sin 5\alpha| + (1/360) |\sin 6\alpha| \\ & \text{(maximum between } 69^\circ 31' \text{ and } 69^\circ 32'), \end{aligned}$$

$$\begin{aligned} \phi_7(\alpha) = & 10.5078 |\sin \alpha| + 3.8056 |\sin 2\alpha| + 0.9305 |\sin 3\alpha| \\ & + 0.1609 |\sin 4\alpha| + (1/40) |\sin 5\alpha| + (1/360) |\sin 6\alpha| + 2/7! \\ & \text{(maximum between } 69^\circ 54' \text{ and } 69^\circ 55'), \end{aligned}$$

$$\begin{aligned}
\phi_8(\alpha) &= 14.4630 |\sin \alpha| + 5.2539 |\sin 2\alpha| + 1.2686 |\sin 3\alpha| \\
&\quad + 0.2327 |\sin 4\alpha| + 0.0322 |\sin 5\alpha| + (1/240) |\sin 6\alpha| \\
&\quad + 2/7! + 2/8! \quad (\text{maximum between } 69^\circ 49' \text{ and } 69^\circ 51'), \\
\phi_9(\alpha) &= 19.926924 |\sin \alpha| + 7.2320 |\sin 2\alpha| + 1.7513 |\sin 3\alpha| \\
&\quad + 0.31714 |\sin 4\alpha| + 0.04653 |\sin 5\alpha| + 0.00537 |\sin 6\alpha| \\
&\quad + 3/7! + 2/8! + 2/9! \quad (\text{maximum between } 69^\circ 49' \text{ and } 69^\circ 51'), \\
\phi_{10}(\alpha) &= 27.4424 |\sin \alpha| + 9.9635 |\sin 2\alpha| + 2.4105 |\sin 3\alpha| \\
&\quad + 0.437825 |\sin 4\alpha| + 0.0634267 |\sin 5\alpha| + 0.0077542 |\sin 6\alpha| \\
&\quad + 3.8598/7! + 3/8! + 2/9! + 2/10! \\
&\quad (\text{maximum between } 69^\circ 49' \text{ and } 69^\circ 51').
\end{aligned}$$

It can be verified by direct computation that

$$(3.2) \quad M_k < 2(1.3775)^{k+1} \quad (k = 1, 2, 3),$$

$$(3.3) \quad M_k < (1.3775)^{k+1} \quad (k = 4, 5, 6, 7, 8, 9, 10),$$

$$(3.4) \quad L_k < (1.3775)^k \quad (k = 1, 2, 3, 4),$$

$$(3.5) \quad L_k < 0.7692(1.3775)^k \quad (k = 5, 6, 7, 8, 9, 10).$$

From (3.1) we have

$$\begin{aligned}
(3.6) \quad nL_n &< M_{n-1} + M_{n-2} + 1.5M_{n-3} + 1.9299M_{n-4} + 2.7915M_{n-5} \\
&\quad + \sum_{r=6}^{n-5} L_{r-1}M_{n-r} + 4.8414L_{n-5} + 3.6379L_{n-4} + 2.5981L_{n-3} \\
&\quad + 2L_{n-2} + L_{n-1}.
\end{aligned}$$

If we assume (3.3) and (3.5) are satisfied also for $11 \leq k \leq n-1$, then (3.6) gives, if we write $\gamma = 1.3775$, $\mu = 0.7692$,

$$\begin{aligned}
nL_n &< \gamma^n + \gamma^{n-1} + 1.5\gamma^{n-2} + 1.9299\gamma^{n-3} + 2.7915\gamma^{n-4} \\
&\quad + \mu[(n-10)\gamma^n + 4.8414\gamma^{n-5} + 3.6379\gamma^{n-4} + 2.5981\gamma^{n-3} \\
&\quad + 2\gamma^{n-2} + \gamma^{n-1}] < n\mu\gamma^n - 0.0005\gamma^{n-5}.
\end{aligned}$$

Hence $L_n < \mu\gamma^n$.

This proves (3.5) is true for all $k \geq 11$, by induction.

From (2.1) for $n \geq 11$,

$$\begin{aligned}
M_n &\leq \max_{0 \leq \alpha \leq \pi/2} \left\{ \sum_{r=1}^6 \frac{2 |\sin r\alpha|}{r!} L_{n-r} \right\} + \sum_{r=7}^n \frac{2}{r!} L_{n-r} \\
&< \mu\gamma^{n-7} \max_{0 \leq \alpha \leq \pi/2} \Phi(\alpha) + \sum_{r=7}^n \frac{2}{r!} \gamma^{7-r}
\end{aligned}$$

where

$$\Phi(\alpha) = \sum_{r=1}^6 \frac{2|\sin r\alpha|}{r!} \gamma^{7-r},$$

which has its maximum between $69^\circ 49'$ and $69^\circ 51'$, giving

$$\max_{0 \leq \alpha \leq \pi/2} \Phi(\alpha) < 16.8520.$$

Hence

$$\begin{aligned} M_n &< 16.8520\mu\gamma^{n-7} + \gamma^{n-7} \left[\frac{2}{7!} + \frac{2}{8!} \frac{1}{\gamma} + \frac{2}{9!} \frac{1}{\gamma^2} + \cdots \right] \\ &< 16.8520\mu\gamma^{n-7} + \gamma^{n-7} \frac{2}{7!} \left[1 + \frac{1}{8\gamma} + \frac{1}{(8\gamma)^2} + \cdots \right] \\ &= 16.8520\mu\gamma^{n-7} + \frac{2\gamma^{n-7}}{7!(1 - 1/8\gamma)} \\ &< \gamma^{n-7} [12.9626 + 0.0006] \\ &< \gamma^{n+1}. \end{aligned}$$

This proves (3.3) for all $k \geq 11$, by induction.

Since G_n is analytic in the z_r it follows that its maximum modulus is assumed when each z_r is on the circumference of the unit circle. Thus we have the following theorem.

THEOREM I. *If z_r is a sequence of points in the unit circle, then*

$$M_n = \max |G_n(z_0; z_1, z_2, \dots, z_n)| < (1.3775)^{n+1} \quad (n \geq 4).$$

4. Now consider the Gontcharoff polynomials for the case discussed by Schoenberg, namely $G_n(x; x_1, x_2, \dots, x_n)$ where

$$-1 \leq x_r \leq +1 \quad (1 \leq r \leq n).$$

Consider any one of the 2^{n-r} polynomials

$$G_n(x; x_1, x_2, \dots, x_r, \pm 1, \pm 1, \dots, \pm 1) \quad (1 \leq r \leq n),$$

$$\frac{\partial G_n}{\partial x_r} = -G_{r-1}(x; x_1, x_2, \dots, x_{r-1}) \times G_{n-r}(x_r, \pm 1, \pm 1, \dots).$$

As x_r varies between -1 and $+1$, keeping $x_1, x_2, \dots, x_{r-1}$ fixed, $\partial G_n / \partial x_r$ is of constant sign, that is, $G_n(x; x_1, \dots, x_r, \pm 1, \pm 1, \dots, \pm 1)$ increases or decreases steadily. Hence $|G_n(x; x_1, \dots, x_r, \pm 1, \pm 1, \dots, \pm 1)|$ attains its maximum when x_r is an end point.

If we take $r=1, 2, \dots, n$, it follows that $|G_n(x; x_1, \dots, x_n)|$ ($-1 \leq x_r$

≤ 1) attains its maximum for any given value of x ($-1 \leq x \leq 1$) when $x_r = \pm 1$ ($1 \leq r \leq n$).

So, in order to find an upper bound for $|G_n(x; x_1, x_2, \dots, x_n)|$ ($-1 \leq x_r \leq 1$), it is sufficient to consider the 2^n polynomials $|G_n(x; \pm 1, \pm 1, \dots, \pm 1)|$ ($-1 \leq x \leq 1$).

Clearly if $0 \leq x \leq 1$ and $x_r = \pm 1$,

$$(4.1) \quad |G_n(x; +1, x_2, \dots, x_n)| = |G_n(-x; -1, -x_2, \dots, -x_n)| \\ \leq |G_n(0; -1, -x_2, \dots, -x_n)|$$

$$(4.2) \quad \leq |G_n(x; -1, -x_2, \dots, -x_n)|.$$

I shall prove that if $0 \leq x \leq 1$ and $x_r = \pm 1$ ($1 \leq r \leq n$) for all n ,

$$(4.3) \quad |G_n(x; x_1, x_2, \dots, x_n)| \leq 2 \left(\frac{4}{\pi} \right)^{n-1} \sin \frac{\pi}{4} (x+1).$$

By (4.2), it is sufficient to prove (4.3) for the case $x_1 = -1$, that is, it is sufficient to prove

$$(4.4) \quad |G_n(x; -1, +1, x_3, \dots, x_n)| \leq 2 \left(\frac{4}{\pi} \right)^{n-1} \sin \frac{\pi}{4} (x+1)$$

and

$$(4.5) \quad |G_n(x; -1, -1, x_3, \dots, x_n)| \leq 2 \left(\frac{4}{\pi} \right)^{n-1} \sin \frac{\pi}{4} (x+1).$$

Proof of (4.4).

$$|G_{n+1}(x; -1, +1, x_3, \dots, x_{n+1})| = \int_{-1}^x |G_n(x'; +1, x_3, \dots, x_{n+1})| dx' \\ = I_1 + I_2,$$

where

$$I_1 = \int_{-1}^0 |G_n(x'; +1, x_3, \dots, x_{n+1})| dx',$$

$$I_2 = \int_0^x |G_n(x'; +1, x_3, \dots, x_{n+1})| dx'.$$

If we use (4.1),

$$I_1 = \int_{-1}^0 |G_n(-x'; -1, -x_3, \dots, -x_{n+1})| dx'.$$

If we substitute $x = -x'$,

$$I_1 = \int_0^1 |G_n(x; -1, -x_3, \dots, -x_{n+1})| dx.$$

Now if we assume that (4.3) is true if n is replaced by any number $m \leq n$,

$$I_1 \leq 2 \left(\frac{4}{\pi} \right)^{n-1} \int_0^1 \sin \frac{\pi}{4} (x+1) dx = 2^{1/2} \left(\frac{4}{\pi} \right)^n,$$

$$\begin{aligned} I_2 &= \int_0^x dx' \int_{x'}^1 |G_{n-1}(x''; x_3, \dots, x_{n+1})| dx'' \\ &\leq 2 \left(\frac{4}{\pi} \right)^{n-2} \int_0^x dx' \int_{x'}^1 \sin \frac{\pi}{4} (x''+1) dx'' \\ &= 2 \left(\frac{4}{\pi} \right)^n \sin \frac{\pi}{4} (x+1) - 2^{1/2} \left(\frac{4}{\pi} \right)^n. \end{aligned}$$

Therefore $I_1 + I_2 \leq 2(4/\pi)^n \sin(\pi/4)(x+1)$.

But (4.4) is true when $n=0, 1$.

Hence (4.4) is true for all n by induction.

Proof of (4.5).

$$\begin{aligned} G_{n+1}(x; -1, -1, x_3, \dots, x_n) &= \int_{-1}^x |G_n(x'; -1, x_3, \dots, x_n)| dx' \\ &= I_3 + I_4 \end{aligned}$$

where

$$\begin{aligned} I_3 &= \int_{-1}^0 |G_n(x'; -1, x_3, \dots, x_n)| dx', \\ I_4 &= \int_0^x |G_n(x'; -1, x_3, \dots, x_n)| dx'. \end{aligned}$$

If we use (4.1),

$$I_3 = \int_{-1}^0 |G_n(-x'; +1, -x_3, \dots, -x_n)| dx'.$$

If we substitute $x = -x'$,

$$\begin{aligned} I_3 &= \int_0^1 |G_n(x; +1, -x_3, \dots, -x_n)| dx \\ &= \int_0^1 \int_x^1 |G_{n-1}(x'; -x_3, \dots, -x_n)| dx' dx. \end{aligned}$$

Hence, if we assume (4.3) is true if n is replaced by any number $m \leq n$,

$$\begin{aligned}
 I_3 &\leq 2 \left(\frac{4}{\pi} \right)^{n-2} \int_0^1 dx \int_x^1 \sin \frac{\pi}{4} (x' + 1) dx' \\
 &= \left(\frac{4}{\pi} \right)^n (2 - 2^{1/2}). \\
 I_4 &\leq 2 \left(\frac{4}{\pi} \right)^{n-1} \int_0^x \sin \frac{\pi}{4} (x' + 1) dx' \\
 &= \left(\frac{4}{\pi} \right)^n \left(-2 \cos \frac{\pi}{4} (x + 1) + 2^{1/2} \right).
 \end{aligned}$$

Now $1 - \cos (\pi/4)(x+1) \leq \sin (\pi/4)(x+1)$, $0 \leq x \leq 1$. Hence $I_3 + I_4 \leq 2(4/\pi)^n \sin (\pi/4)(x+1)$.

But (4.5) is true when $n=0, 1$. Hence (4.5) is true for all n by induction.

Since (4.4) and (4.5) are true, we have proved (4.3). It follows by substituting $-x$ for x , that for $-1 \leq x \leq 0$ and $-1 \leq x_r \leq 1$ ($1 \leq r \leq n$),

$$(4.6) \quad |G_n(x; x_1, x_2, \dots, x_n)| \leq 2 \left(\frac{4}{\pi} \right)^{n-1} \cos \frac{\pi}{4} (x + 1),$$

and we have the following theorem.

THEOREM II. *If z_r is a sequence of points on the real axis, satisfying $-1 \leq z_r \leq 1$, then*

$$(4.7) \quad |G_n(z_0; z_1, z_2, \dots, z_n)| \leq 2 (4/\pi)^{n-1}.$$

5. I shall now discuss extensions of Theorems I and II in which some of the points of the sequence z_r lie outside the unit circle, and the segment $-1 \leq x \leq 1$ respectively.

Let $z_r = x_r + y_r$, where both x_r and y_r may be complex, then since $G_n(z_0; z_1, \dots, z_n)$ is a polynomial in each z_r ($0 \leq r \leq n$), we may apply Taylor's series and write

$$(5.1) \quad G_n(z_0; z_1, \dots, z_n) = \exp \left(\sum_{r=0}^n y_r \frac{\partial}{\partial x_r} \right) G_n(x_0; x_1, \dots, x_n).$$

Now, writing $G_n(x_0; x_1, x_2, \dots, x_n) = G_n$, using (2.2) and (2.3), we note that $\partial G_n / \partial x_r$ ($0 \leq r \leq n$) is either one multiple integral or the product of two such integrals, in each case the total multiplicity being $n-1$. Similarly $\partial^k G_n / \partial x_r \partial x_s \dots \partial x_t$, where $r, s, \dots, t$ may all take any values between 0 and n inclusive, is either zero (for example, $\partial^2 G_n / \partial x_0 \partial x_1$) or the product of not more than $k+1$ multiple integrals, the total multiplicity being $n-k$.

Now suppose that positive constants A, γ can be found such that

$$(5.2) \quad |G_n| < A \gamma^{n+1}$$

provided that the sequence $\{x_r\}$ belongs to a given set of points S which in-

cludes $z=0$. Such a set exists by Theorem I.

Setting $n=0$, we see that $A\gamma > 1$. Hence

$$\left| \frac{\partial^k G_n}{\partial x_r \partial x_s \cdots \partial x_t} \right| < A^{k+1} \gamma^{n+1}.$$

Suppose also that the values of y_r are restricted in such a way that

$$(5.3) \quad \sum_{r=1}^n |y_r| \leq nh$$

for certain values of n . Then (5.1) gives, for these values of n ,

$$(5.4) \quad \begin{aligned} |G_n(z_0; z_1, z_2, \dots, z_n)| &< \sum_{k=0}^{\infty} \frac{(|y_0| + nh)^k}{k!} A^{k+1} \gamma^{n+1} \\ &= A\gamma^{n+1} \exp \{A(|y_0| + nh)\}. \end{aligned}$$

If the sequence $\{z_r\}$ is such that all its limit points belong to S , then (5.3) is satisfied for arbitrarily small h and sufficiently large n , and (5.4) gives

$$(5.5) \quad |G_n(z_0; z_1, \dots, z_n)| < A e^{A|y_0|} (\gamma + \epsilon)^{n+1}, \quad n \geq n_0(\epsilon),$$

and hence for all z in any given finite domain, and all n ,

$$(5.6) \quad |G_n(z; z_1, z_2, \dots, z_n)| < A'(\gamma + \epsilon)^{n+1}.$$

6. Suppose now that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an integral function satisfying

$$\limsup_{r \rightarrow \infty} \frac{\log M(r)}{r} = \sigma < \frac{1}{\gamma},$$

it follows that for any $\tau > \sigma$, and sufficiently large n

$$(6.1) \quad n! |a_n| < \tau^n.$$

Then, if $f(z_1) = 0$, $f^{(n-1)}(z_n) = 0$, clearly

$$f(z) = \int_{z_1}^z dz' \int_{z_2}^{z'} dz'' \cdots \int_{z_n}^{z^{(n-1)}} f^{(n)}(z) dz,$$

or, following Levinson [3, §1], if we replace $f^{(n)}(z)$ by its power series, we obtain

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} (n+k)! \frac{a_{n+k}}{k!} \int_{z_1}^z dz' \int_{z_2}^{z'} dz'' \cdots \int_{z_n}^{z^{(n-1)}} z^k dz \\ &= \sum_{k=0}^{\infty} (n+k)! a_{n+k} G_{n+k}(z; z_1, z_2, \dots, z_n, 0, 0, \dots, 0). \end{aligned}$$

Now since the sequence $\{z_n\}$ is such that all its limit points belong to S , then for large n and for all z in any finite domain we have by (5.6) and (6.1)

$$(6.2) \quad |f(z)| < \sum_{k=0}^{\infty} \tau^{n+k} A'(\gamma + \epsilon)^{n+k+1} = \frac{A'(\gamma + \epsilon)^{n+1} \tau^n}{1 - \tau(\gamma + \epsilon)}$$

provided $\tau < 1/(\gamma + \epsilon)$. But letting $n \rightarrow \infty$ in (6.2) we have $f(z) \equiv 0$.

In the particular case in which S is the unit circle, Theorem I shows that (5.2) is satisfied with $\gamma = 1.3775 < 1/.7259$ for all values of n , so we now have the following theorem.

THEOREM III. *If $f(z)$ is an integral function satisfying*

$$\limsup_{r \rightarrow \infty} \frac{\log M(r)}{r} < .7259,$$

and if $f(z_1) = 0, f^{(n-1)}(z_n) = 0$ ($n \geq 2$), the sequence $\{z_r\}$ having all its limit points in the unit circle, then $f(z) \equiv 0$.

In the particular case in which S is the segment $0 \leq x \leq 1$, Theorem II shows that (5.2) is satisfied for all n with $\gamma = 4/\pi$ and we have the following extension of Schoenberg's theorem.

THEOREM IV. *If $f(z)$ is an integral function satisfying*

$$\limsup_{r \rightarrow \infty} \frac{\log M(r)}{r} < \frac{\pi}{4},$$

and if $f(z_1) = 0, f^{(n-1)}(z_n) = 0$ ($n \geq 2$), the sequence $\{z_r\}$ having all its limit points on the segment $-1 \leq x \leq 1$ of the real axis, then $f(z) \equiv 0$.

This result has been stated by Kamenetsky [2, Theorem VIII] but I have been unable to find a published proof. It seems unlikely from the context that his method has anything in common with the one which I have used here.

7. A further theorem follows as a consequence of inequalities (5.2) and (5.6) for the case in which the limit points of the sequence of zeros lie inside the locus of points distant h from the segment $-1 \leq x \leq 1$ of the real axis. We shall call the domain enclosed by this curve H . In this case, if we restrict the sequence $\{x_r\}$ to the segment $-1 \leq x \leq 1$ (all r) and $z_r = x_r + y_r$ (all $r \geq 1$) where $|y_r| \leq h$ ($r \geq 1$), (5.2) is satisfied with $A = \pi^2/8$, $\gamma = 4/\pi$, by Theorem II, and (5.3) is satisfied for all n since $|y_r| \leq h$ ($r \geq 1$). Hence (5.4) is satisfied for all n with these values of the constants, that is,

$$|G_n(z_0; z_1, z_2, \dots, z_n)| \leq \frac{\pi^2}{8} \left(\frac{4}{\pi}\right)^{n+1} \exp \left\{ \frac{\pi^2}{8} (|y_0| + nh) \right\} < \bar{A} \tilde{\gamma}^{n+1},$$

with $\tilde{\gamma} = (4/\pi) \exp(\pi^2 h/8)$. By a second application of formulae (5.2) and (5.6), we see that, provided all the limit points of the sequence $\{z_r\}$ lie within H , (5.6) holds with $\gamma = (4/\pi) \exp(\pi^2 h/8)$, and we have the following theorem.

THEOREM V. If $f(z)$ is an integral function satisfying

$$\limsup_{r \rightarrow \infty} \frac{\log M(r)}{r} < \frac{\pi}{4} \exp \left(-\frac{\pi^2 h}{8} \right),$$

and if $f(z_1) = 0$, $f^{(n-1)}(z_n) = 0$ ($n \geq 2$), where the sequence $\{z_r\}$ has all its limit points in H , then $f(z) \equiv 0$.

It is to be noted that the constant $(\pi/4) \exp(-\pi^2 h/8)$ is "better" (that is, greater) than that obtained from the circle circumscribed to H , namely, $.7259/1+h$ (which is obtained from Theorem III by the transformation $\zeta = (1+h)z$) only for small values of h . It is "better" when $h \leq 0.23$ but not when $h = 0.24$.

APPENDIX

Upper bounds for

n	M_{n-1}	L_n	$L_n/(1.3775)^n$	$(1.3775)^n$
1	1	1	0.7260	1.3775
2	2	1.5	0.7905	1.8975
3	2.5981	1.9299	0.7384	2.6138
4	3.6379	2.7915	0.7753	3.6005
5	4.8414	3.8056	0.7673	4.9597
6	6.8223	5.2539	0.7690	6.8320
7	9.3973	7.2315	0.7685	9.4111
8	12.9512	9.9635	0.7686	12.9638
9	17.8413	13.7212	0.7684	17.8577
10	24.5754	18.8998	0.7683	24.5989
11	33.8472			33.8850

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